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Perturbation of the Zeros of Analytic Functions. I

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1. INTRODUCTION

We study here the variation of the zeros of an analytic function under perturbation of the function. Our main questions are:

(1) If g is close to f, how close are the zeros of g to those of f?

(2) What is the first variation of the zeros of f?

To answer the latter question, we seek formulas for the zeros of g, with error of the magnitude $o(|g-f|)$.

Of course, in order to answer these questions, we must describe the closeness of g to f in terms of a suitable topology. For local questions it is sufficient to work with the Banach algebra R of bounded analytic functions in the unit circle U, with the norm

$$\|f\| = \sup\{|f(z)| \mid z \in U\}.$$

A classical theorem of Hurwitz [1] says that if $f_n \to f$ uniformly on every compact subset of U and $f \not\equiv 0$, then the limit points of the zeros of f_n are precisely the zeros of f. This tells us that the zeros of f are continuous functions of f with respect to the weak topology of R, so long as we exclude a weak neighborhood of 0. A basis for the weak neighborhoods of 0 is the family of sets $U(z_0, \epsilon)$ of the form

$$U(z_0, \epsilon) = \{f \mid |f(z_0)| < \epsilon\}.$$

The neighborhoods of 0 in the weak topology are the unions of the finite intersections of sets $U(z_0, \epsilon)$. So it is sufficient to study the first question in the complement of a set $U(z_0, a)$.

We are thus led to the problem:

Given $z_0 (0 < |z_0| < 1)$, $a > 0$ and $\epsilon > 0$, find a $\delta > 0$ such that if $f, g \in R$,

$$\|f\| \leqslant 1, \qquad |f(z_0)| \geqslant a, \qquad f(0) = 0,$$

and

$$\|g - f\| \leqslant \delta,$$

then g has a zero in the circle $U_\epsilon : |z| < \epsilon$.

This is equivalent to the problem:

Given z_0, a, and ϵ as above, find $\delta > 0$ such that if $g \in R$, and

$$\|g\| \leqslant 1, \qquad |g(z_0)| \geqslant a, \qquad |g(0)| < \delta,$$

then g has a zero in U_ϵ.

In other words, we must show that if $g \in R$, $\|g\| \leqslant 1$, and $|g(z_0)| \geqslant a$, then near any point where g is approximately zero, there is a point where g is exactly zero. We give an answer to this question in

THEOREM 1. *If* $g \in R$, $\|g\| \leqslant 1, 0 < s < 1$,

$$M(s) = \max\{|g(z)| \mid |z| = s\} > 0,$$

and

$$\lambda = \lambda(s) = \log(1/M(s))/\log(1/s),$$

and if

$$|g(0)| \leqslant \delta \leqslant M(s)\exp(-4\lambda),$$

then g has a zero in the circle U_ϵ, *where*

$$\epsilon = 4e^2 \lambda^{-1} \delta^{1/\lambda} \log(1/\delta).$$

This result is nontrivial ($\epsilon < 1$) if $\delta < a^\lambda$, where a is the solution of the equation

$$a\log(1/a) = 1/4e^2, \qquad 0 < a \leqslant 1/e.$$

In many applications it is more convenient to use a different normalization, with the roles of the points 0 and z_0 interchanged. We are thus led to the problem:

given $\qquad M > 1, \qquad 0 < \alpha \leqslant 1, \qquad 0 < r < 1,$

find $\delta > 0$ such that if

$f \in R, \qquad f(0) = 1, \qquad \|f\| \leqslant M, \qquad$ and $\qquad |f(z_0)| < \delta, \qquad$ where $|z_0| = r$,

then f has a zero in the circle $|z - z_0| < \alpha(1 - r)$.

We obtain

THEOREM 2. *If* $f \in R, f(0) = 1, \|f\| \leqslant M$, *and*

$$|f(z_0)| \leqslant \delta = \exp(-K\lambda),$$

where $|z_0| = r = e^{-x}, 0 < r < 1$,

$$\lambda = \log M/\log(1/r),$$

and $K \geqslant 7$, *then f has a zero in the circle*

$$|z - z_0| \leqslant \{12e^2(x + K)\exp(-x - K)\}(1 - r).$$

One application is a quantitative form of the well-known fact that the zeros of an analytic function of several variables are not isolated.

THEOREM 3. *If f is analytic in the bicylinder* $U \times U$, $\|f\| \leqslant 1$,

$$f(0,0) = 0, \qquad M = \max\{|f(z,0)| \mid |z| = s\} > 0,$$

and

$$T = 4e(|t|/a(s))^{1/\lambda} < s/e,$$

where

$$\lambda = \log(1/M)/\log(1/s)$$

and

$$a(s) = \frac{M\log(1/s)}{2e + M\log(1/s)},$$

then $f(z,t)$ *has a zero in the circle*

$$|z| \leqslant eT\log(1/T).$$

A corollary makes quantitative the fact that the mapping by a nonconstant analytic function is open:

COROLLARY 3a. *If* $F \in R$, $\|F\| \leqslant 1$, $M = M(s) = \max\{|F(z)| \mid |z| = s\}$ *and* $F(0) = 0$, *then the image of the circle* U_s *under the mapping* F *contains the circle* U_r *where* $r = a(s)(s/4e^2)^\lambda$.

Another application (see Theorem 4 below) is to the estimate of the minimum modulus of an analytic function f on the complement in U of the union of the open disks with centers at the zeros z_k of f and with radii $\alpha(1 - |z_k|)$.

If F is an entire function, $F(0) = 1$,

$$M(R) = \max\{|F(z)| \mid |z| = R\},$$

and

$$|F(z_0)| \leqslant \exp(-K\lambda_2(r)), \qquad |z_0| = r,$$

where

$$\lambda_2(r) = \min\{\log M(R)/\log(R/r) \mid R > r\},$$

and $K \geqslant 7$, then F has a zero in the circle

$$|z - z_0| \leqslant Hr,$$

where

$$H = 4e^2(1 - B^{-1})(K + x)\, e^{-K},$$

rB is the smallest R for which

$$\log M(R) = \lambda_2(r)\log(R/r),$$

and $B = e^x$.

For example, if $M(R) \leqslant A\exp(CR^k)$ for $R \geqslant r$, where $k > 0$, if $r^k \geqslant \log A/eC$ and if

$$|F(z_0)| \leqslant \exp(-Ke^2 kCr^k), \qquad K \geqslant 7, \qquad |z_0| \leqslant r,$$

then F has a zero in the circle

$$|z - z_0| \leqslant 8e^2 k^{-2}(2 + kK)\, e^{-K} r.$$

If W is an integral operator,

$$(Wx)(t) = \int_0^1 W(t,s)x(s)\,ds,$$

on the space $C[0,1]$ of continuous functions on the unit interval, the kernel $W(t,s)$ is continuous, and

$$\tau = \tau(W) = \max\{|W(t,s)| \mid 0 \leqslant t \leqslant 1, 0 \leqslant s \leqslant 1\},$$

then the resolvent of W can be expressed as the quotient of two entire functions of order 2, whose growth can be expressed in terms of τ. The above results enable us to show that if

$$\|x - \lambda_0 Wx\| \leqslant \epsilon, \qquad \|x\| = 1, \qquad |\lambda_0| \leqslant r,$$

and ϵ is sufficiently small, then λ_0 is near an eigenvalue of W. In fact, if

$$\epsilon \leqslant 7^{-1} \exp(-(2e^2 K + 1)\tau^2 r^2),$$

$K \geqslant 7$, and $\tau r \geqslant 1/2e$, then W has an eigenvalue in the circle

$$|\lambda - \lambda_0| \leqslant 4e^2(1 + K)e^{-K} r.$$

This result is valid, more generally, if W is an operator of trace class on any Banach space S.

The above results give estimates for the continuity of the zeros of f which are uniform in any bounded subset of R outside of a neighborhood $U(z_0,a)$. No assumption is made on the order of a zero of f. If f has a zero of known order at a point z_0, then we can obtain more precise information about the zeros of a nearby function g.

In the second part of this paper we study the variation of a single zero, of given order, of the function $f \in R$, under small perturbation. The case $f(z) = z^n$, $n \geqslant 1$, yields a simple proof of the Weierstrass preparation theorem. Our approach also yields a quantitative form of this theorem.

In the third part of the paper we study the simultaneous variation of all the zeros of f in a compact subset of U, under small perturbation of f. These results yield estimates for an analytic function of a matrix or, more generally, of an algebraic element of a Banach algebra. We also obtain the beginnings of an elimination theory for analytic functions.

2. Perturbation Outside of a Weak Neighborhood of Zero

Our first problem is:

Given a, s, and r in the open interval $(0,1)$, find $\delta = \delta(a,s,r)$ such that if $g \in R$, $\|g\| \leqslant 1$, $|g(z_0)| \geqslant a$, where $|z_0| = s$, and $|g(0)| < \delta$, then g has a zero in $U_r: |z| < r$.

Let $g \in R$,

$$M(s) = \max\{|g(z)| \mid |z| = s\}, \qquad 0 < s < 1,$$

and

$$Z(g) = \sup\{r \mid g(z) \neq 0 \text{ in } U_r\}.$$

Then the problem is clearly equivalent to that of estimating $Z(g)$ in terms of s, $M(s)$, and $|g(0)|$. By a classical result (see [1], p. 183), we know that for $s = \rho Z(g)$, $0 < \rho < 1$,

$$\log M(s) \leqslant \left(\frac{1-\rho}{1+\rho}\right) \log |g(0)|. \tag{1}$$

If $|g(0)| = M(s)^A$, then we obtain

$$\rho \geqslant \frac{A-1}{A+1}$$

and

$$Z(g) \leqslant \left(\frac{A+1}{A-1}\right) s. \tag{2}$$

This gives us a bound less than 1 if

$$A > \frac{1+s}{1-s}.$$

Thus, if

$$\delta \leqslant \exp\left(\frac{1+s}{1-s} \log M(s)\right)$$

and $|g(0)| < \delta$, then $Z(g) < 1$, and we have the bound (2) for $Z(g)$, where

$$A = \frac{\log |g(0)|}{\log M(s)} > \frac{\log \delta}{\log M(s)}.$$

As $\delta \to 0+$, the above bound for $Z(g)$ approaches s, while we seek a bound which approaches 0 as $\delta \to 0+$.

If we use the bound (1) for $M(\rho r)$, $r = Z(g)$, and apply the Hadamard 3-circle theorem to estimate $M(s)$, assuming that $\rho r < s < 1$, we obtain

$$\log M(s) < \frac{\log(1/s)}{\log(1/\rho r)} \left(\frac{1-\rho}{1+\rho}\right) \log \delta,$$

or

$$\log(1/\delta) < \lambda \log(1/\rho r) \left(\frac{1+\rho}{1-\rho}\right), \tag{3}$$

where

$$\lambda = \lambda(s) = \frac{\log(1/M(s))}{\log(1/s)}.$$

Of course, by the convexity of $\log M(s)$ as a function of $\log s$, we know that $\lambda(s)$ is a nondecreasing function, and if $M(1) = \|g\| = 1$, we have

$$\lim_{s \to 1} \lambda(s) = M'(1).$$

For $0 < r \leqslant 1$, let

$$\eta(r) = \min_{0<\rho<1} \left(\frac{1+\rho}{1-\rho}\right) \log(1/r\rho),$$

and let $\rho(r)$ be the ρ for which the minimum is attained. We conclude that if $r\rho(r) \leqslant s$, then

$$\eta(r) \geqslant \lambda^{-1} \log(1/\delta).$$

Since η is a decreasing function of r, we obtain

$$Z(g) = r \leqslant \eta^{-}{}_{1}(\lambda^{-1} \log(1/\delta)), \tag{4}$$

where η^{-1} is the (decreasing) inverse function of η.

The bound (4) is an estimate of $Z(g)$ in terms of s, $M(s)$, and δ, and since, as we shall show, $\eta(0+) = +\infty$ (i.e., $\eta^{-1}(\infty) = 0$), this bound will serve our purpose.

To study η, we set $\rho = 1/x$, $y = \log(1/r)$, so that $x > 1$, $y > 0$. Then we have

$$\eta = \min_{x>1} K(x),$$

where

$$K(x) = \left(\frac{x+1}{x-1}\right)(y + \log x).$$

The minimum of K is attained at the solution of the equation

$$F(x) = \frac{x^2-1}{2x} - \log x = y.$$

In the appendix it is shown that F has an increasing inverse function G, $G(0) = 1$, and that

$$2y + 2\log(y+1) + 1 \leqslant G(y) \leqslant 2y + 2\log(y+1) + 1 + 4\log 2.$$

Since

$$\eta = \frac{(x+1)^2}{2x} = \frac{x}{2} + 1 + \frac{1}{2x}, \qquad \text{for } x = G(y),$$

we find that

$$y + \log(y+1) + 3/2 \leqslant \eta \leqslant y + \log(y+1) + 2 + 2\log 2,$$

or

$$\log(e/r) + \log_2(e/r) + 1/2 \leqslant \eta \leqslant \log(e/r) + \log_2(e/r) + 1 + 2\log 2.$$

We wish to find a sufficient condition that $r\rho(r) \leqslant s$. This is equivalent to ($y = \log(1/r)$, $x = G(y)$)

$$\log(1/s) \leqslant y + \log x = (x/2) - 1/(2x).$$

Since

$$(x/2) - 1/(2x) \geqslant y + \log(y+1) + \tfrac{1}{2}(1 - 1/x) \geqslant y + \log(y+1),$$

it is sufficient that

$$\log(e/s) \leqslant y + 1 + \log(1 + y),$$

or[1]

$$p(\log(e/s)) \leqslant y + 1 = \log(e/r).$$

If we apply the estimate (*) of the Appendix, we find that it is sufficient that

$$r \leqslant s(1 + \log(e/s))/2.$$

The estimate (2) above shows that this will be true if

$$|g(0)| \leqslant \delta_1(s) = M(s)\exp(-4\lambda(s)).$$

It this is true, then the above estimates certainly hold, and it follows that

$$\eta - \log\eta - 1 - 2\log 2 \leqslant \log(e/r),$$

and

$$r \leqslant 4e^2\lambda^{-1}\delta^{1/\lambda}\log(1/\delta).$$

We have thus proved

THEOREM 1. *If* $g \in R$, $\|g\| \leqslant 1$, $0 < s < 1$, *and if*

$$|g(0)| \leqslant \delta \leqslant \delta_1(s) = M(s)\exp(-4\lambda(s)),$$

where

$$M(s) = \max(|g(z)| \mid |z| = s),$$

and

$$\lambda(s) = \log(1/M(s))/\log(1/s),$$

then

$$Z(g) \leqslant 4e^2\lambda^{-1}\delta^{1/\lambda}\log(1/\delta).$$

The above estimates yield this alternative formulation which is sometimes simpler to use:

COROLLARY 1a. *If* $g \in R$, $\|g\| \leqslant 1$, $r, s \in (0,1)$, *if*

$$r/\log(e/r) \leqslant s,$$

and

$$|g(0)| < \left(\frac{r}{4e^2\log(e/r)}\right)^{\lambda(s)},$$

then $Z(g) < r$.

We now interchange the roles of the point where g is small and the one where g is not too small. This leads to the problem:

Given $M > 0$, $0 < r < 1$, and $\epsilon > 0$, find α such that whenever

$$f \in R, \quad f(0) = 1, \quad \|f\| \leqslant M, \quad \text{and} \quad |f(z_0)| < \epsilon, \quad \text{where } |z_0| = r,$$

then f has a zero in the circle $|z - z_0| < \alpha(1 - r)$.

[1] $p(t)$ is the solution of $n + \log n = t$.

Let

$$g(z) = f(\zeta)/M,$$

where

$$\zeta = (z + z_0)/(1 + \bar{z}_0 z),$$

so that

$$g(0) = f(z_0)/M, \qquad g(-z_0) = 1/M.$$

Then if

$$M(r) = \max\{|g(z)| \mid |z| = r\},$$

we have

$$\lambda = \log(1/M(r))/\log(1/r),$$

$$\lambda \leqslant \log M/\log(1/r) = \lambda_1 = 1/\mu.$$

Theorem 1 implies that if

$$\epsilon \leqslant \exp(-4\lambda_1),$$

then

$$Z(g) \leqslant 4e^2 a \log(1/a) = b,$$

where $a = r\epsilon^{\mu}$.

The circle $|z| \leqslant b$ is mapped into the circle

$$|\zeta - z_0| \leqslant \frac{(1 - r^2)b}{1 - rb} \leqslant 3b(1 - r)$$

if $b \leqslant 1/3$. Now, $b \leqslant 1/3$ if and only if $a \leqslant \exp(-c)$, where c is the solution of the equation $c - \log c = \log(4e^2)$, and this is equivalent to

$$\epsilon \leqslant M \exp(-c\lambda_1).$$

Hence, if

$$\epsilon \leqslant \min(\exp(-4\lambda_1), M\exp(-c\lambda_1)),$$

then f has a zero in the circle $|z - z_0| \leqslant 3b(1 - r)$. We note that $M\exp(-c\lambda_1) \leqslant \exp(-4\lambda_1)$ if and only if $r \geqslant \exp(4 - c)$, and that c is about 6.4. For many purposes, the following result is adequate:

THEOREM 2. *If $f \in R$, $\|f\| = M$, $f(0) = 1$ and*

$$|f(z_0)| \leqslant \epsilon \leqslant \exp(-7\lambda_1),$$

where $|z_0| = r$, $\lambda_1 = \log M/\log(1/r)$, then f has a zero in the circle $|z - z_0| \leqslant 3b(1 - r)$, where b is defined as above.

It is well known that the zeros of an analytic function of several variables are not isolated. We can apply the above results to obtain quantitative information about the zeros.

Suppose, for example, that f is analytic in the bicylinder $U \times U$, $\|f\| \leqslant 1$, $f(0,0) = 0$ and

$$M(s,t) = \max\{|f(z,t)| \mid |z| = s\} > 0 \qquad \text{for } t = 0.$$

Let

$$Z(t) = \sup\{r \mid f(z,t) \neq 0 \text{ for } |z| < r \leqslant 1\}.$$

We wish to estimate $Z(t)$ in terms of $|t|$, s, and $M(s,0)$.

Let

$$g(t)(z) = f(z,t).$$

Then for $t \in U$, $g(t) \in R$ and $\|g(t)\| \leqslant 1$. By the Schwarz lemma, applied to $(f(z,t) - f(z,0))/2$ for fixed z, we obtain

$$\|g(t) - g(0)\| \leqslant 2|t|,$$

so that

$$|g(t)(0)| = |f(0,t)| \leqslant 2|t|$$

and

$$|M(s,t) - M(s,0)| \leqslant 2|t|.$$

Therefore, we have

$$\lambda(s,t) \leqslant \frac{4|t|}{M \log(1/s)}$$

if $|t| \leqslant M/4$, $M = M(s,0)$.

By Corollary 1a, if $0 < r < 1$,

$$r/\log(e/r) < s,$$

and

$$2|t| < A^{-\lambda(s,t)}, \qquad \text{where } A = 4e^2 \log(e/r)/r,$$

then $Z(t) < r$.

But if

$$|t| \leqslant \frac{M \log(1/s)}{4 \log A},$$

then

$$\lambda(s,t) \log A \leqslant \lambda \log A + 1, \qquad \lambda = \lambda(s,0),$$

and then we have

$$2|t| A^{\lambda(s,t)} \leqslant 2e|t| A^{\lambda} \leqslant 1$$

if

$$|t| \leqslant A^{-\lambda}/2e.$$

Consequently, it is sufficient that

$$|t| \leqslant A^{-\lambda} a(s),$$

where

$$a(s) = \frac{M \log(1/s)}{2e + M \log(1/s)}.$$

Here we have used the fact that

$$A \geqslant 4e^2/s \geqslant 4e^2$$

and that $x^{-\lambda}\log x$ is decreasing for $x \geqslant \exp(1/\lambda)$, which implies that

$$A^{-\lambda}\log A \leqslant (4e^2)^{-\lambda}\log(4e^2)$$
$$\leqslant (4e^2)^{-1}\log(4e^2) < 1.$$

We have thus proved

THEOREM 3. *If f is analytic in $U \times U$, $\|f\| \leqslant 1$, and*

$$f(0,0) = 0, \qquad M = \max\{|f(z,0)| \mid |z| = s\} > 0,$$

then for

$$A = 4e^2\log(e/r)/r > 4e^2/s$$

and

$$|t| \leqslant a(s)A^{-\lambda},$$

we have $Z(t) \leqslant r$.

Here

$$\lambda = \log(1/M)/\log(1/s),$$

and $a(s)$ is given above.

If we set

$$|t|A^{\lambda} = a(s),$$

then the above conditions are satisfied if

$$T = 4e(|t|/a(s))^{1/\lambda} < s/e \tag{5}$$

and

$$\log(e/r) + \log_2(e/r) = \log(1/T).$$

The estimate, in the Appendix, of the solution of the latter equation yields

COROLLARY 3a. *Under the hypotheses of the theorem, if (5) holds, then*

$$Z(t) \leqslant eT\log(1/T),$$

and therefore

$$\lim_{|t|\to 0} \frac{\log(1/Z(t))}{\log(1/|t|)} \geqslant 1/\lambda.$$

By the Puiseux expansion, we know that as $t \to 0$,

$$Z(t) \sim b|t|^c,$$

where $b \neq 0$, and c is a positive rational number. The exact determination of c requires, in general, rather detailed information about the coefficients of the power series for f. Our corollary implies that $c \geqslant 1/\lambda$, and this estimate for c requires only a lower bound on $f(z_0, 0)$ for a single z_0.

If we apply this corollary to $f(z,t) = (F(z) - t)/2$, where $F \in R$, $\|F\| \leqslant 1$, and $F(0) = 0$, we find a constant $c(s)$ such that the range of F contains the circle: $|t| < c(s)$. A somewhat better and simpler estimate for $c(s)$ is obtained by applying the above reasoning to the function $g(t) = (F - t)/(1 + |t|)$.

COROLLARY 3b. *If $F \in R$, $\|F\| \leqslant 1$, $\max\{|F(z)| \mid |z| = s\} = M > 0$, and $F(0) = 0$, then the image of the circle U_s: $|z| < s$ under the mapping F contains the circle U_r, $r = a(s)(s/4e^2)^\lambda$.*

Suppose that $f \in R$, $\|f\| \leqslant M$, $f(0) = 1$, and let $0 < \alpha \leqslant 1$. Let $E_\alpha = U - C_\alpha$, where C_α is the union of the disks of radii $\alpha(1 - |z_k|)$ about each zero z_k of f in U. We wish to estimate

$$m_\alpha(r) = \min\{|f(z)| \mid |z| = r, z \in E_\alpha\}$$

from below.

If $|z| = r$, and $|f(z)| \leqslant \epsilon \leqslant M\exp(-7\lambda_1)$, where

$$\lambda_1 = \log M/\log(1/r),$$

then there is a zero ζ of f such that

$$|z - \zeta| < 3b\,(1 - r),$$

where

$$b = 4e^2 a \log(1/a), \qquad a = r\epsilon^\mu, \qquad \mu = 1/\lambda_1.$$

But $3b(1 - r)$ will be less than $\alpha(1 - |\zeta|)$ if $6b < \alpha$. This will be true if

$$\log(1/a) \geqslant J(\log\beta),$$

where $\beta = 24e^2/\alpha$, and $J(x)$ is the solution of

$$y - \log y = x.$$

By the estimate $J(x) \leqslant x + \log x + 1$, given in the Appendix, we find that $z \in C_\alpha$ if

$$\epsilon \leqslant M\exp(-\lambda_1 \log(e\beta\log\beta)).$$

We thus obtain

THEOREM 4. *If $f \in R$, $\|f\| \leqslant M$, $f(0) = 1$, and $0 < \alpha \leqslant 1$, and if E_α* and $m_\alpha(r)$ *are defined as above, then*

$$m_\alpha(r) > M\exp(-\lambda_1 \log(e\beta\log\beta)),$$

where

$$\lambda_1 = \log M/\log(1/r)$$

and $\beta = 24e^2/\alpha$.

If we allow α to depend on $|\zeta|$, say $\alpha = (1 - |\zeta|)^k$, then we can still obtain similar results on $m_\alpha(r)$.

Suppose that F is an entire function, $F(0)=1$, and $M(R)=\max\{|F(z)| \mid |z|=R\}$. We apply Theorem 2 to $f(z)=F(Rz)$, and find that if $|z_0|=r<R$, and

$$|F(z_0)| \leqslant \epsilon \leqslant \exp(-7\lambda_1),$$

where

$$\lambda_1=\lambda_1(R)=\log M(R)/\log(R/r),$$

then F has a zero in the circle

$$|z-z_0| \leqslant 3b(R-r),$$

where

$$b=4e^2 a\log(1/a), \qquad a=r\epsilon^{\mu}/R, \qquad \mu=1/\lambda_1.$$

Let

$$\lambda_2(r)=\min_{R>r} \lambda_1(R),$$

and let $rB(r)$ be the smallest R for which this minimum is attained. Then if $\epsilon \leqslant \exp(-7\lambda_2(r))$, the function F has a zero in the circle

$$|z-z_0| \leqslant 4e^2(B-1)\,ra\log(1/a), \qquad \text{where } a=\epsilon^{\mu}/B.$$

Thus, if $M(R) \leqslant A\exp(CR^k)$ for $R \geqslant r$, $k>0$, then the favorable choice is $R=rB(r)$, $B(r)=e^x$, where x is the solution of the equation

$$(kx-1)\exp(kx)=\log A/Cr^k=e\delta.$$

We set $kx=1+y$, so that y satisfies

$$ye^y=\delta.$$

For $\delta \geqslant -1/e$, there is a unique solution $y \geqslant -1$, and if $\delta \geqslant 0$ we have $0 \leqslant y \leqslant \delta$. (Incidentally, if $-1/e \leqslant \delta \leqslant 0$, we have $e\delta \leqslant y \leqslant \delta$.)

For $\lambda_2(r)$ we obtain the estimate

$$\lambda_2(r) \leqslant kCr^k\exp(1+y) \leqslant ekCr^k e^{\delta}=\lambda_3(r).$$

We note that

$$1 \leqslant M(r) \leqslant A\exp(Cr^k),$$

which implies that $\delta \geqslant -1/e$.

A little computation yields

THEOREM 5. *If F is an entire function, $F(0)=1$, and $M(R) \leqslant A\exp(CR^k)$ for $R \geqslant r$, $k>0$, and if $|z_0|=r$ and*

$$|F(z_0)| \leqslant \epsilon \leqslant \exp(-KekCr^k e^{\delta}), \qquad K \geqslant 7,$$

where

$$\delta=\log A/eCr^k,$$

then F has a zero in the circle

$$|z-z_0| \leqslant 4e^2 k^{-2}(1+\delta)(1+\delta+kK)\,e^{-K}r.$$

In many cases, the resolvent of a linear operator W may be expressed as a quotient of two entire functions of known growth. For example, if W is an integral operator on $C[0,1]$ with a continuous kernel:

$$(Wx)(t) = \int_0^1 W(t,s)x(s)\,ds,$$

or more generally, if W is an operator of trace class (see Zaanen [4], Smithies [3], Ruston [2]) on any Banach space S, or if W is an operator of the Hilbert–Schmidt class (see Smithies) on Hilbert space, the resolvent

$$R(\lambda) = (I-\lambda W)^{-1} = D(\lambda)/d(\lambda),$$

where $d(\lambda)$ is a scalar valued, and $D(\lambda)$ is an operator valued, entire function of order 2.

In such cases, we can prove that if $x - \lambda Wx = y$ is sufficiently small, where $\|x\| = 1$, $|\lambda| = r$, then λ is close to an eigenvalue. For more general W, such estimates do not seem likely to be true. The method is to note that from the equation

$$d(\lambda)\,x = D(\lambda)\,y$$

we can obtain the estimate

$$|d(\lambda)| \leqslant \|D(\lambda)\|\,\|y\| \leqslant \epsilon\|D(\lambda)\|.$$

Making use of the known estimates of d and D, we can apply Theorem 5.

For example, if W is of trace class with trace-norm $\tau = \tau(W)$, then we can obtain

$$d(\lambda) \leqslant \exp(\tau r + \tau^2 r^2/2) \leqslant \exp(1/2 + \tau^2 r^2)$$

for $|\lambda| = r$, and

$$\begin{aligned}\|D(\lambda)\| &\leqslant |d(\lambda)| + \|\lambda W D(\lambda)\| \\ &\leqslant |d(\lambda)| + (\tau r)^{1/2} \exp((3\tau r + \tau^2 r^2)/2) \\ &\leqslant 7\exp(\tau^2 r^2).\end{aligned}$$

(These estimates are somewhat better than those of Ruston, but can easily be obtained by modifications of his method.)

An easy computation, using Theorem 5, now gives

THEOREM 6. *If W is a linear operator of trace class on a Banach space S, of trace-norm τ, and if*

$$x \in S, \qquad \|x\| = 1, \qquad |\lambda_0| = r, \qquad \tau r \geqslant 1/2e,$$

and

$$\|x - \lambda_0 Wx\| \leqslant \epsilon \leqslant 7^{-1}\exp(-(2e^2K+1)\tau^2 r^2),$$

where $K \geqslant 7$, then there is an eigenvalue of W in the circle

$$|\lambda - \lambda_0| \leqslant 4e^2(1+K)\,e^{-K} r.$$

It would be interesting to have an estimate of the distance from x to an eigenvector.

In these estimates, no effort was made to determine how close they are to the best possible. It would be desirable, for many applications to numerical analysis, to know whether at least the orders of magnitude are correct.

Appendix

We need to obtain information about the solution x of the equation

$$F(x) = \frac{x^2 - 1}{2x} - \log x = y, \qquad y > 0, x > 1.$$

Since

$$F'(x) = \frac{(x-1)^2}{2x},$$

we see that F is an increasing function and has an increasing inverse function G, $G(0) = 1$.

The above equation can be written in the form

$$x = 2y + 2\log x + 1/x.$$

Since

$$\frac{d}{dx}(2\log x + 1/x) = (2x-1)/x^2 > 0 \qquad \text{for } x > 1/2,$$

we see that

$$x > 2y + 1.$$

Set $x = 2X - 1$, $X = (x+1)/2 > 1$. Then we have

$$\begin{aligned} X &= y + 1/2 + \log(2X-1) + 1/[2(2X-1)] \\ &= Y + \log X, \end{aligned}$$

where

$$Y = y + 1/2 + \log(2 - 1/X) + 1/[2(2X-1)].$$

Note that the sum of the last two terms is an increasing function of X for $X \geqslant 1$, so that

$$1/2 \leqslant \log(2 - 1/X) + 1/[2(2X-1)] \leqslant \log 2$$

and

$$y + 1 \leqslant Y \leqslant y + 1/2 + \log 2.$$

We are led to study the solution X of the equation

$$H(X) = X - \log X = Y,$$

for $X \geqslant 1$, $Y \geqslant 1$. Since

$$H'(X) = 1 - 1/X > 0 \qquad \text{for } X > 1,$$

H is increasing and has an increasing inverse function $J(Y)$, and $J(1) = 1$.

We have

$$X = Y + \log X \geqslant Y + \log Y,$$

and

$$X \leqslant Y + \log Y + \log(X/Y)$$
$$\leqslant Y + \log Y + (X/Y) - 1,$$

so that

$$X = Y + \log Y + \log Y/(Y-1)$$
$$< Y + \log Y + 1 \qquad \text{for } Y > 1.$$

Thus, we have

$$Y + \log Y \leqslant J(Y) \leqslant Y + \log Y + 1.$$

We apply this estimate to $x = 2X - 1$, use the estimates for Y in terms of y, and obtain

$$2y + 2\log(y+1) + 1 \leqslant x \leqslant 2y + 2\log(y+1) + 1 + 4\log 2.$$

From

$$\eta = (x/2) + 1 + 1/2x,$$

we obtain

$$y + \log(y+1) + 3/2 \leqslant \eta \leqslant y + \log(y+1) + 2 + 2\log 2.$$

To estimate y in terms of η, we examine the equation

$$h(n) = n + \log n = r, \qquad n \geqslant 1, \qquad r \geqslant 1.$$

The function h has an increasing inverse function p. If $n \geqslant 1$, $r \geqslant 1$, we have

$$r \geqslant n$$

and therefore

$$r \leqslant n + \log r, \qquad \text{or } n \geqslant r - \log r.$$

It follows that

$$r \geqslant n + \log(r - \log r)$$
$$\geqslant n + \log r + \log(1 - \log r/r)$$
$$\geqslant n + \log r - \log r/(r - \log r).$$

But the maximum of $\log r/r$ for $r > 1$ is $1/e$, so that

$$\log r/(r - \log r) \leqslant (e-1)^{-1}.$$

Therefore

$$r - \log r \leqslant n \leqslant r - \log r + (e-1)^{-1}.$$

Since

$$\eta - 1 - 2\log 2 \leqslant h(y+1) \leqslant \eta - 1/2,$$

we obtain

$$\eta - 1 - 2\log 2 - \log(\eta - 1 - 2\log 2) \leqslant y + 1$$
$$\leqslant \eta - 1/2 - \log(\eta - 1/2) + (e-1)^{-1}.$$

For some purposes, a simpler estimate of $p(r)$ is more useful:

$$r \leqslant 2n - 1, \qquad \text{i.e., } n \geqslant (r+1)/2,$$

from which we obtain

$$p(r) = n = r - \log n \leqslant r - \log (r+1)/2). \tag{*}$$

This is cruder than the above for large r, but is quite good for r close to 1.

We remark that if $1 \leqslant w \leqslant n + \log n + C$ and $1 \leqslant n$, then

$$w - \log w - C \leqslant n.$$

If $w \leqslant C + 1$, this is trivial. If $w \geqslant C + 1$, it follows from

$$w - C - \log(w - C) \leqslant p(w - C) \leqslant n.$$

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